BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 35(3), 395-418

Concentration phenomena in weakly coupled elliptic systems with critical growth*

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Abstract. In this paper we consider the weakly coupled elliptic system with critical growth

$$\begin{cases} -\Delta u = |u|^{\frac{4}{N-2}} u + \varepsilon\big[a(x)u + b(x)v\big] & \text{in } \Omega, \\ -\Delta v = |v|^{\frac{4}{N-2}} v + \varepsilon\big[c(x)u + d(x)v\big] & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where a, b, c, d are C^1- functions defined in a bounded regular domain Ω of $\mathbb{R}^N$. Here we construct families of solutions which blow-up and concentrate at some points in Ω as the positive parameter ε goes to zero.

Keywords: elliptic systems, critical nonlinearity, Dirichlet boundary condition.

Mathematical subject classification: 35J60.

1 Introduction and main results

In this paper we consider the weakly coupled elliptic system

$$\begin{cases} -\Delta u_1 = |u_1|^{p-1}u_1 + \varepsilon\big[a(x)u_1 + b(x)u_2\big] & \text{in } \Omega, \\ -\Delta u_2 = |u_2|^{p-1}u_2 + \varepsilon\big[c(x)u_1 + d(x)u_2\big] & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$, $N \geq 5$, $\varepsilon > 0$, $p = \frac{N+2}{N-2}$, $a, b, c, d \in C^1(\bar{\Omega})$.

Received 2 September 2004.

*The authors are supported by M.I.U.R., project “Metodi variazionali e topologici nello studio di fenomeni non lineari”.

If $b(x) = c(x)$ in Ω system (1.1) is of gradient-type, namely (1.1) is the Euler-Lagrange equation of a suitable functional defined on the space $\mathrm{H}^1_0(\Omega) \times \mathrm{H}^1_0(\Omega)$.

In [2] the authors consider the scalar case, namely a, b, c, d are real constants. Using a variational argument they proved that if the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is symmetric, so that the system is of gradient-type, and $a > 0$ or $d > 0$ then for ε small enough there exists a nontrivial weak solution of (1.1). Moreover, using a Pohozaev-type identity (see [15]), they prove that if A is symmetric and negative definite and Ω is star-shaped $u_1, u_2 = 0$ is the unique classical solution to (1.1).

If $b(x) \neq c(x)$ then system (1.1) is not variational. As it is pointed out in [7] it seems that the only available technique to treat such systems is topological, explicitely the topological degree of Leray-Schauder. But in this case one needs a priori bounds on the positive solutions of (1.1). How to get such bounds can be seen in [7], where an extensive list of references is given on this matter. In particular in the case of a weakly coupled system a priori bound exists in the subcritical case (see [1]).

Here we are interested in studying the weakly coupled system (1.1) in the critical case. In particular we want to find solutions which concentrate in some points of Ω in the sense of the following definition.

Definition 1.1. *Let* $(u_{1_\varepsilon}, u_{2_\varepsilon})$ *be a family of solutions for* (1.1). *We say that* $(u_{1_\varepsilon}, u_{2_\varepsilon})$ *blow-up and concentrate at the points* ξ_1 *and* ξ_2 *in* Ω *if there exist rates of concentration* $\delta_{1\varepsilon}, \delta_{2\varepsilon}$ *and points* $\xi_{1\varepsilon}, \xi_{2\varepsilon} \in \Omega$ *with* $\lim_{\varepsilon\to 0} \delta_{i\,\varepsilon} = 0$ *and* $\lim_{\varepsilon\to 0} \xi_{i\,\varepsilon} = \xi_i$ *such that* $u_{i_\varepsilon} - P_\Omega U_{\delta_{i\,\varepsilon}, \xi_{i\,\varepsilon}}$, $i = 1, 2$, *go to zero in* $H^1_0(\Omega)$ *as* ε *goes to zero.*

Here (see [3], [6] and [17])

$$U_{\lambda,y}(x) = C_N \frac{\lambda^{\frac{N-2}{2}}}{\left(\lambda^2 + |x-y|^2\right)^{\frac{N-2}{2}}}, \quad x \in \mathbb{R}^N,\ y \in \mathbb{R}^N,\ \lambda > 0,$$

with $C_N = [N(N-2)]^{(N-2)/4}$, are all the positive solutions of the problem $-\Delta U = U^{\frac{N+2}{N-2}}$ in $\mathbb{R}^N$. $P_\Omega U_{\lambda,y}$ denotes the projection onto $\mathrm{H}^1_0(\Omega)$ of $U_{\lambda,y}$, i.e. $\Delta P_\Omega U_{\lambda,y} = \Delta U_{\lambda,y}$ in Ω, $P_\Omega U_{\lambda,y} = 0$ on $\partial\Omega$.

Before to state our results we need to introduce some notation.

Let us denote by G the Green's function of the negative laplacian on Ω and by H its regular part, chosen in such a way that

$$H(x,y) = \frac{B_N}{|x-y|^{N-2}} - G(x,y), \quad \forall (x,y) \in \Omega^2,$$

where $B_N = \left[(N-2)\,\text{meas}(S^{N-1})\right]^{-1}$ and S^{N-1} is the $(N-1)-$dimensional unit sphere. For every $x \in \Omega$ the leading term of H, namely $\tau(x) := H(x,x)$, is called Robin function of Ω at the point x. The *harmonic radius* r is defined by $r(x) = \tau(x)^{-\frac{1}{N-2}}$. It is a smooth positive function in the interior of Ω, which vanishes at every point on the boundary.

Blowing-up solutions appear in a large class of problems with critical growth. For example, as far as it concerns the Brezis-Nirenberg problem (see [5])

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} + \varepsilon a(x) u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

it was proved that (see [16] and [13]) any "stable" critical point ξ_0 of the function $\Psi(\xi) = a(\xi) r(\xi)^2$ with $a(\xi_0) > 0$ generates a family of solutions to (1.2) which blow-up and concentrate at ξ_0.

In this paper we first consider the case of different concentration points, namely in Definition 1.1 it holds $\xi_1 \neq \xi_2$. Let us introduce the functions $\Psi_1, \Psi_2 : \Omega \to \mathbb{R}$ defined by $\Psi_1(\xi) = a(\xi) r(\xi)^2$ and $\Psi_2(\xi) = d(\xi) r(\xi)^2$.

Definition 1.2. *Let $\Psi : \Omega \to \mathbb{R}$ be a C^1-function, we say that ξ_0 is a stable critical point of Ψ if $\nabla\Psi(\xi_0) = 0$ and there exists a neighbourhood $V \subset\subset \Omega$ of ξ_0 such that $\nabla\Psi(\xi) \neq 0 \quad \forall\, \xi \in \partial V$, if $\nabla\Psi(\xi) = 0$, $\xi \in V$, then $\Psi(\xi) = \Psi(\xi_0)$ and* deg $\left(\nabla\Psi, \bar{V}, 0\right) \neq 0$, *where* deg *denotes the Brouwer degree.*

Notice that any isolated local maximum point or any isolated local minimum point or any nondegenerate critical point of Ψ are stable critical point of Ψ.

Theorem 1.3. *Let $N \geq 5$. For $i = 1, 2$ let ξ_i be a stable critical point of Ψ_i with $\Psi_i(\xi_i) > 0$. If $\xi_1 \neq \xi_2$, then there exists a family of solutions of problem* (1.1) *that blow-up and concentrate at two points ξ_1^* and ξ_2^* such that $\nabla\Psi_i(\xi_i^*) = 0$ and $\Psi_i(\xi_i^*) = \Psi_i(\xi_i)$, with rates of concentration $\delta_{i\,\varepsilon}$ such that*

$$\lim_{\varepsilon \to 0} \delta_{i\,\varepsilon} \varepsilon^{-\frac{1}{N-4}} = \left(\frac{2}{N-2} \frac{B}{A^2} \Psi_i(\xi_i^*) \right)^{\frac{1}{N-4}} r(\xi_i^*)$$

(see Lemma 4.1 *and Lemma* 4.2*).*

It is not difficult to show examples in which Theorem 1.3 applies (see examples 4.7 and 4.8).

If we consider the case when concentration points are the same, namely in Definition 1.1 it holds $\xi_1 = \xi_2$, the problem becomes much more difficult of the previous one. We are only able to treat the symmetric case, where we can assume the crucial condition $\xi_{1_\varepsilon} = \xi_{2_\varepsilon} = 0$. It remains open the case when the concentration points are the same, but $\xi_{1_\varepsilon} \neq \xi_{2_\varepsilon}$.

In Section 5 we assume that Ω is a symmetric domain, namely for any $i = 1, \dots, N$

$$(x_1, \dots, x_i, \dots, x_N) \in \Omega \quad \Longleftrightarrow \quad (x_1, \dots, -x_i, \dots, x_N) \in \Omega. \tag{1.3}$$

We say that a function $w : \Omega \to \mathbb{R}$ is symmetric if for any $i = 1, \dots, N$

$$w(x_1, \dots, x_i, \dots, x_N) = w(x_1, \dots, -x_i, \dots, x_N). \tag{1.4}$$

We prove the following result.

Theorem 1.4. *Let Ω be a symmetric domain and a, b, c, d be symmetric functions. Assume one of the following conditions*

(1) $N \geq 5$, $a(0) = d(0) = 0$ *and* $b(0), c(0) > 0$;

(2) $N \geq 5$, $a(0), d(0) > 0$ *and* $b(0), c(0) \geq 0$;

(3) $N \geq 7$, $a(0), d(0) > 0$ *and* $b(0), c(0) \leq 0$.

Then there exists a family of symmetric solutions of problem (1.1) *that concentrates at the origin.*

We would like to emphasize the fact that using Theorem 5.6 and Proposition 5.8, we can find more general conditions on $a(0)$, $b(0)$, $c(0)$ and $d(0)$ which ensure the existence of families of blowing-up solutions. At this aim we quote Example 5.9 where a non-uniqueness result is proved (see also Theorem 5.4 and Remark 5.10).

We want to point out that the solutions given in Theorems 1.3 and 1.4 are actually positive if the system is cooperative, namely $b(x), c(x) \geq 0$ in Ω (see Proposition 4.6).

Finally we remark that if Ω is symmetric with respect to the origin (i.e. $x \in \Omega$ iff $-x \in \Omega$) we can construct solutions which are symmetric with respect to the origin (i.e. $w(x) = w(-x)$) provided assumptions (1), (2) or (3) of Theorem 1.4 are satisfied (see Remark 5.11).

The paper is organized as follows. In Section 2 we set the problem in a suitable framework and in Section 3 we reduce the problem to a finite dimensional one using a Ljapunov-Schmidt reduction argument as in [4] and [10]. This tool allows us to treat both variational and not variational system. In Section 4 we study the finite dimensional problem and we prove Theorem 1.3. Section 5 deals with the symmetric case and with the proof of Theorem 1.4.

2 Setting of the problem

Let $\alpha = \frac{1}{N-4}$ and set $\Omega_\varepsilon = \Omega/\varepsilon^\alpha$. An easy computation shows that, if $u_1(x)$, $u_2(x)$ are solutions of (1.1), then $v_1(y) = \varepsilon^{\alpha\frac{N-2}{2}} u_1(\varepsilon^\alpha y)$, $v_2(y) = \varepsilon^{\alpha\frac{N-2}{2}} u_2(\varepsilon^\alpha y)$ solve

$$\begin{cases} -\Delta v_1 = |v_1|^{p-1} v_1 + \varepsilon^{2\alpha+1}\Big[a(\varepsilon^\alpha y) v_1 + b(\varepsilon^\alpha y) v_2\Big] & \text{in } \Omega_\varepsilon, \\ -\Delta v_2 = |v_2|^{p-1} v_2 + \varepsilon^{2\alpha+1}\Big[c(\varepsilon^\alpha y) v_1 + d(\varepsilon^\alpha y) v_2\Big] & \text{in } \Omega_\varepsilon, \\ v_1 = v_2 = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{2.5}$$

Let $\mathrm{H}_0^1(\Omega_\varepsilon)$ be the Hilbert space equipped with the usual inner product $(u, v)_{\mathrm{H}_0^1} = \int\limits_{\Omega_\varepsilon} \nabla u \nabla v$, which induces the norm $\|u\|_{\mathrm{H}_0^1} = \left(\int\limits_{\Omega_\varepsilon} |\nabla u|^2\right)^{1/2}$.

Moreover, if $r \in [1, +\infty)$ and $u \in L^r(\Omega_\varepsilon)$, we will set $\|u\|_r = \left(\int\limits_{\Omega_\varepsilon} |u|^r\right)^{1/r}$.

It will be useful to rewrite problem (2.5) in a different setting. Let us then introduce the following operator.

Definition 2.1. *Let $i_\varepsilon^* : L^{\frac{2N}{N+2}}(\Omega_\varepsilon) \longrightarrow H_0^1(\Omega_\varepsilon)$ be the adjoint operator of the immersion $i_\varepsilon : H_0^1(\Omega_\varepsilon) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega_\varepsilon)$, i.e.*

$$i_\varepsilon^*(u) = v \quad \Longleftrightarrow \quad (v, \varphi) = \int\limits_{\Omega_\varepsilon} u(x)\varphi(x)dx \quad \forall\, \varphi \in \mathrm{H}_0^1(\Omega_\varepsilon).$$

Remark 2.2. There exists $c > 0$ such that

$$\|i_\varepsilon^*(u)\|_{\mathrm{H}_0^1} \leq c\|u\|_{\frac{2N}{N+2}} \qquad \forall\, u \in \mathrm{L}^{\frac{2N}{N+2}}(\Omega_\varepsilon), \quad \forall\, \varepsilon > 0.$$

Let $\mathrm{H} = \mathrm{H}_0^1(\Omega_\varepsilon) \times \mathrm{H}_0^1(\Omega_\varepsilon)$, which is an Hilbert space equipped with the inner product $\Big((u_1, u_2), (\phi_1, \phi_2)\Big)_{\mathrm{H}} = (u_1, \phi_1)_{\mathrm{H}_0^1} + (u_2, \phi_2)_{\mathrm{H}_0^1}$ that induces the norm

$$\|(u_1, u_2)\| = \left(\|u_1\|_{\mathrm{H}_0^1}^2 + \|u_2\|_{\mathrm{H}_0^1}^2\right)^{1/2}.$$

For $(u_1, u_2) \in \mathrm{H}$ and $r \in \left[1, \frac{2N}{(N-2)}\right]$, we set $\|(u_1, u_2)\|_r = \|u_1\|_r + \|u_2\|_r$.

By Remark 2.2 we get the following result.

Lemma 2.3. *Let $\mathfrak{I}_\varepsilon^* : L^{\frac{2N}{N+2}}(\Omega_\varepsilon) \times L^{\frac{2N}{N+2}}(\Omega_\varepsilon) \longrightarrow H$ be defined by $\mathfrak{I}_\varepsilon^*(u_1, u_2) = \Big(i_\varepsilon^*(u_1), i_\varepsilon^*(u_2)\Big)$.*

Then $\mathfrak{I}^*_\varepsilon$ is continuous uniformly with respect to ε, namely there exists $c > 0$ such that

$$\left\|\mathfrak{I}^*_\varepsilon(u_1, u_2)\right\| \le S^{-\frac{1}{2}} \|(u_1, u_2)\|_{\frac{2N}{N+2}}, \ \forall\, u_1, u_2 \in \mathrm{L}^{\frac{2N}{N+2}}(\Omega_\varepsilon), \ \forall\, \varepsilon > 0.$$

By means of the definition of the operator $\mathfrak{I}^*_\varepsilon$, problem (2.5) turns out to be equivalent to

$$(u_1, u_2) = \mathfrak{I}^*_\varepsilon\left(F(u_1, u_2) + \varepsilon^{2\alpha+1} G(\varepsilon^\alpha y, u_1, u_2)\right), \quad u \in \mathrm{H}, \tag{2.6}$$

where

$$\begin{aligned} F(s,t) &= (f(s), f(t)), \ f(s) = |s|^{p-1}s \quad \text{and} \\ G(x,s,t) &= (a(x)s + b(x)t, c(x)s + d(x)t). \end{aligned}$$

We are looking for solutions $(u_1(x), u_2(x))$ to (2.6) of the form

$$(u_1(x), u_2(x)) = \left(P_\varepsilon U_{\lambda_1, \xi_1/\varepsilon^\alpha}(x) + \phi_{1\varepsilon}(x), P_\varepsilon U_{\lambda_2, \xi_2/\varepsilon^\alpha}(x) + \phi_{2\varepsilon}(x)\right),$$

where we have denoted $P_\varepsilon = P_{\Omega_\varepsilon}$. Here $\phi_\varepsilon(x) = \left(\phi_{1\varepsilon}(x), \phi_{2\varepsilon}(x)\right)$ is a lower order term belonging to a suitable subspace of H which will be introduced in the following.

Let us denote

$$\psi^0_{\lambda,y}(x) = \frac{\partial U_{\lambda,y}}{\partial \lambda} = C_N \frac{N-2}{2} \lambda^{\frac{N-4}{2}} \frac{|x-y|^2 - \lambda^2}{(\lambda^2 + |x-y|^2)^{N/2}}, \qquad x \in \mathbb{R}^N,$$

and for $j = 1, \dots, N$

$$\psi^j_{\lambda,y}(x) = \frac{\partial U_{\lambda,y}}{\partial y_j} = -C_N(N-2)\lambda^{\frac{N-2}{2}} \frac{x_j - y_j}{(\lambda^2 + |x-y|^2)^{N/2}}, \quad x \in \mathbb{R}^N.$$

The space spanned by $\psi^j_{\lambda,y}$, $j = 0, 1, \dots, N$, is the set of the solutions of the linearized problem $-\Delta\psi = pU^{p-1}_{\lambda,y}\psi$, in $\mathbb{R}^N$. Moreover let

$$P_\varepsilon \psi^j_{\lambda,\xi/\varepsilon^\alpha}(x) = i^*_\varepsilon\left(U^{p-1}_{\lambda,\xi/\varepsilon^\alpha} \psi^j_{\lambda,\xi/\varepsilon^\alpha}\right)(x) \qquad x \in \Omega_\varepsilon. \tag{2.7}$$

For $i = 1, 2$, let $K^i_\varepsilon = \text{span}\langle P_\varepsilon \psi^0_{\lambda_i,\xi_i/\varepsilon^\alpha}, P_\varepsilon \psi^1_{\lambda_i,\xi_i/\varepsilon^\alpha}, \dots, P_\varepsilon \psi^N_{\lambda_i,\xi_i/\varepsilon^\alpha}\rangle$ and

$$K^{i\,\perp}_\varepsilon = \left\{\phi \in H^1_0(\Omega) : (\phi, P_\varepsilon \psi^j_{\lambda_i,\xi_i/\varepsilon^\alpha}) = 0, \ j = 0, 1, \dots, N\right\}.$$

Moreover let us define the operators

$$\Pi_\varepsilon^i(u) = \sum_{j=0}^{N} (u, P_\varepsilon \psi^j_{\lambda,\xi/\varepsilon^\alpha})_{\mathrm{H}_0^1} P_\varepsilon \psi^j_{\lambda,\xi/\varepsilon^\alpha} \quad \text{and} \quad \Pi_\varepsilon^{i,\perp}(u) = u - \Pi_\varepsilon^i(u).$$

Set $\lambda = (\lambda_1, \lambda_2)$ and $\xi = (\xi_1, \xi_2)$ and let us consider the subspace of H given by $K_{\varepsilon,\lambda,\xi} = K_\varepsilon^1 \times K_\varepsilon^2$ and its complementary space $K^\perp_{\varepsilon,\lambda,\xi} = K_\varepsilon^{1\perp} \times K_\varepsilon^{2\perp}$.

Finally let us introduce the operators $\Pi_{\varepsilon,\lambda,\xi} : \mathrm{H} \longrightarrow K_{\varepsilon,\lambda,\xi}$ and $\Pi^\perp_{\varepsilon,\lambda,\xi} : \mathrm{H} \longrightarrow K^\perp_{\varepsilon,\lambda,\xi}$, defined by $\Pi_{\varepsilon,\lambda,\xi}(u_1, u_2) = \left(\Pi^1_\varepsilon(u_1), \Pi^2_\varepsilon(u_2)\right)$ and $\Pi^\perp_{\varepsilon,\lambda,\xi}(u_1, u_2) = (u_1, u_2) - \Pi_{\varepsilon,\lambda,\xi}(u_1, u_2)$. If $\mu \in (0,1)$, we set

$$\begin{aligned} \mathcal{O}_\mu = \Big\{ &(\lambda, \xi) \in \mathbb{R}^2 \times \Omega^2 : \lambda_i \in (\mu, \mu^{-1}), \\ &\operatorname{dist}(\xi_i, \partial\Omega) \geq \mu, i = 1, 2, \ |\xi_1 - \xi_2| \geq \mu \Big\}. \end{aligned}$$

By (2.7) and Remark 2.2 we easily deduce the following result:

Lemma 2.4. *For any $\mu \in (0,1)$ there exist $\varepsilon_0 > 0$ and $c > 0$ such that $\|\Pi^\perp_{\varepsilon,\lambda,\xi}(u_1, u_2)\| \leq c\|(u_1, u_2)\|$, for any $(\lambda, \xi) \in \mathcal{O}_\mu$, $\varepsilon \in (0, \varepsilon_0)$ and $(u_1, u_2) \in H$.*

Our approach to solve problem (2.6) will be to find $(\lambda, \xi) \in \mathcal{O}_\mu$, for some μ, and $(\phi_1, \phi_2) \in K^\perp_{\varepsilon,\lambda,\xi}$ such that

$$\begin{aligned} \Pi^\perp_{\varepsilon,\lambda,\xi} \Big\{ &\big(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha} + \phi_1, P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha} + \phi_2\big) \\ &- \mathcal{I}^*_\varepsilon \Big[F\big(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha} + \phi_1, P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha} + \phi_2\big) \\ &+ \varepsilon^{2\alpha+1} G\big(\varepsilon^\alpha y, P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha} + \phi_1, P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha} + \phi_2\big)\Big]\Big\} = 0 \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \Pi_{\varepsilon,\lambda,\xi} \Big\{ &\big(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha} + \phi_1, P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha} + \phi_2\big) \\ &- \mathcal{I}^*_\varepsilon \Big[F\big(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha} + \phi_1, P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha} + \phi_2\big) \\ &+ \varepsilon^{2\alpha+1} G\big(\varepsilon^\alpha y, P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha} + \phi_1, P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha} + \phi_2\big)\Big]\Big\} = 0. \end{aligned} \tag{2.9}$$

3 Finite dimensional reduction

In this section we will solve equation (2.8).

Let us introduce the linear operator $L_{\varepsilon,\lambda,\xi} : K^\perp_{\varepsilon,\lambda,\xi} \to K^\perp_{\varepsilon,\lambda,\xi}$, defined by

$$L_{\varepsilon,\lambda,\xi}(\phi) = \phi - \Pi^\perp_{\varepsilon,\lambda,\xi}\, \mathcal{I}^*_\varepsilon \left(F' \left(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha}(x), P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha}(x)\right) \phi \right).$$

Lemma 3.1. *For any $\mu \in (0,1)$ there exists $\bar{\varepsilon}_1 > 0$ and a constant $C > 0$ such that, for every $(\lambda, \xi) \in \mathcal{O}_\mu$ and for every $\varepsilon \in (0, \bar{\varepsilon}_1)$, the operator $L_{\varepsilon,\lambda,\xi}$ is invertible and it holds $\|L_{\varepsilon,\lambda,\xi}\phi\| \geq C\|\phi\|$ for any $\phi \in K^{\perp}_{\varepsilon,\lambda,\xi}$.*

Proof. The claim follows exactly as in Proposition 3.2 in [13], because

$$L_{\varepsilon,\lambda,\xi}(\phi_1, \phi_2) = (\phi_1 - \Pi^1_\varepsilon[i^*_\varepsilon(f'(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha})\phi_1)],\\ \phi_2 - \Pi^2_\varepsilon[i^*_\varepsilon(f'(P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha})\phi_2)]). \qquad \square$$

By using the invertibility of the operator $L_{\varepsilon,\lambda,\xi}$ we can solve equation (2.8).

Proposition 3.2. *For any $\mu \in (0,1)$ there exist $R, \varepsilon_0 > 0$ such that for every $(\lambda, \xi) \in \mathcal{O}_\mu$ and for any $\varepsilon \in (0, \varepsilon_0)$ there exists a unique $\phi_{\varepsilon,\lambda,\xi} = \left(\phi_{1\varepsilon,\lambda,\xi}, \phi_{2\varepsilon,\lambda,\xi}\right) \in K^{\perp}_{\varepsilon,\lambda,\xi}$ such that*

$$\begin{aligned}\Pi^{\perp}_{\varepsilon,\lambda,\xi}\Big\{&\left(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha} + \phi_1, P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha} + \phi_2\right)\\ &- \mathcal{I}^*_\varepsilon\Big[F\left(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha} + \phi_1, P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha} + \phi_2\right)\\ &+ \varepsilon^{2\alpha+1}G\left(\varepsilon^\alpha y, P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha} + \phi_1, P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha} + \phi_2\right)\Big]\Big\} = 0.\end{aligned} \tag{3.10}$$

Moreover

$$\|\phi_{\varepsilon,\lambda,\xi}\| \leq \begin{cases} R\varepsilon^{\frac{N+2}{2(N-4)}} & \text{if } N \geq 7,\\ R\varepsilon^2|\log\varepsilon| & \text{if } N = 6,\\ R\varepsilon^3 & \text{if } N = 5.\end{cases} \tag{3.11}$$

Proof. First of all we point out that ϕ solves equation (3.10) if and only if ϕ is a fixed point of the operator $T_{\varepsilon,\lambda,\xi} : K^{\perp}_{\varepsilon,\lambda,\xi} \longrightarrow K^{\perp}_{\varepsilon,\lambda,\xi}$ defined by

$$\begin{aligned}T_{\varepsilon,\lambda,\xi}(\phi) = \; & L^{-1}_{\varepsilon,\lambda,\xi}\Pi^{\perp}_{\varepsilon,\lambda,\xi}\mathcal{I}^*_\varepsilon\Big[F(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha} + \phi_1, P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha} + \phi_2)\\ &-F(U_{\lambda_1,\xi_1/\varepsilon^\alpha}, U_{\lambda_2,\xi_2/\varepsilon^\alpha}) - F'(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha}, P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha})(\phi_1, \phi_2)\\ &+\varepsilon^{2\alpha+1}G(\varepsilon^\alpha y, P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha} + \phi_1, P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha} + \phi_2)\Big].\end{aligned}$$

We will show that

$$T_{\varepsilon,\lambda,\xi} : \{\phi \in K^{\perp}_{\varepsilon,\lambda,\xi} : \|\phi\| \leq R\varepsilon^\gamma\} \longrightarrow \{\phi \in K^{\perp}_{\varepsilon,\lambda,\xi} : \|\phi\| \leq R\varepsilon^\gamma\}$$

is a contraction mapping, for R and γ suitable chosen, provided ε is small enough.

From Lemma 2.3, Lemma 2.4 and Lemma 3.1 we get the estimate

$$\begin{aligned} &\|T_{\varepsilon,\lambda,\xi}\phi\| \\ &\quad\le c\,\Big\|F(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha}+\phi_1,\,P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha}+\phi_2) \\ &\qquad - F(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha},\,P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha}) \\ &\qquad - F'(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha},\,P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha})(\phi_1,\phi_2)\Big\|_{\frac{2N}{N+2}} \\ &\qquad + c\,\Big\|F(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha},\,P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha}) - F(U_{\lambda_1,\xi_1/\varepsilon^\alpha},\,U_{\lambda_2,\xi_2/\varepsilon^\alpha})\Big\|_{\frac{2N}{N+2}} \\ &\qquad + c\varepsilon^{2\alpha+1}\Big\|G(\varepsilon^\alpha y,\,P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha}+\phi_1,\,P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha}+\phi_2)\Big\|_{\frac{2N}{N+2}}. \end{aligned} \tag{3.12}$$

First of all we have

$$\begin{aligned} &\|F(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha}+\phi_1,\,P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha}+\phi_2) - F(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha},\,P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha}) \\ &\qquad - F'(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha},\,P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha})(\phi_1,\phi_2)\|_{\frac{2N}{N+2}} \\ &\quad = c\sum_{i=1,2}\|f(P_\varepsilon U_i+\phi_i) - f(P_\varepsilon U_i) - f'(P_\varepsilon U_i)\phi_i\|_{\frac{2N}{N+2}} \\ &\quad \le c\|\phi\|^{\min\{2,p\}}. \end{aligned} \tag{3.13}$$

Secondly, by Lemma 5.3 in [13], we get

$$\begin{aligned} &\Big\|G(\varepsilon^\alpha y,\,P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha}+\phi_1,\,P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha}+\phi_2)\Big\|_{\frac{2N}{N+2}} \\ &\quad \le \max\{\|a\|_\infty,\|c\|_\infty\}\left(\|P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha}\|_{\frac{2N}{N+2}} + \|\phi_1\|_{\frac{2N}{N+2}}\right) \\ &\qquad + \max\{\|b\|_\infty,\|d\|_\infty\}\left(\|P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha}\|_{\frac{2N}{N+2}} + \|\phi_2\|_{\frac{2N}{N+2}}\right) \\ &\quad \le c\left(\chi_2(\varepsilon) + \varepsilon^{-2\alpha}\|\phi\|_{\frac{2N}{N-2}}\right). \end{aligned} \tag{3.14}$$

Finally by Lemma 5.2 in [13] we get

$$\begin{aligned} &\Big\|F(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha},\,P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha}) - F(U_{\lambda_1,\xi_1/\varepsilon^\alpha},\,U_{\lambda_2,\xi_2/\varepsilon^\alpha})\Big\|_{\frac{2N}{N+2}} \\ &\quad = \sum_{i=1,2}\Big\|f(P_\varepsilon U_{\lambda_i,\xi_i/\varepsilon^\alpha}) - f(U_{\lambda_i,\xi_i/\varepsilon^\alpha})\Big\|_{\frac{2N}{N+2}} \le \chi_1(\varepsilon). \end{aligned} \tag{3.15}$$

By (3.13), (3.14), (3.15) we deduce that if $\|\phi\| \le R\varepsilon^\gamma$ as in (3.11), since $\alpha = \frac{1}{N-4}$, then

$$\|T_{\varepsilon,\lambda,\xi}\phi\| \le c\left(\|\phi\|^{\min\{2,p\}} + \chi_1(\varepsilon) + \chi_2(\varepsilon)\varepsilon^{2\alpha+1} + \varepsilon\|\phi\|\right) \le R\varepsilon^\gamma. \tag{3.16}$$

Our next goal is to show that we, actually, have a contraction. Indeed, for any $\phi^1,\ \phi^2 \in K^{\perp}_{\varepsilon,\lambda,\xi}$, we have:

$$\begin{aligned}
&\|T_{\varepsilon,\lambda,\xi}\phi^2 - T_{\varepsilon,\lambda,\xi}\phi^1\| \\
&\quad \le c \sum_{i=1,2} \Big\| f(P_\varepsilon U_{\lambda_i,\xi_i/\varepsilon^\alpha} + \phi_i^2) - f(P_\varepsilon U_{\lambda_i,\xi_i/\varepsilon^\alpha} + \phi_i^1) \\
&\qquad - f'(P_\varepsilon U_{\lambda_i,\xi_i/\varepsilon^\alpha})(\phi_i^2 - \phi_i^1)\Big\|_{\frac{2N}{N+2}} \\
&\qquad + c\varepsilon^{2\alpha+1}\Big\| G(\varepsilon^\alpha y,\, P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha} + \phi_1^2,\, P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha} + \phi_2^2) \\
&\qquad - G(\varepsilon^\alpha y,\, P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha} + \phi_1^1,\, P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha} + \phi_2^1)\Big\|_{\frac{2N}{N+2}}.
\end{aligned} \tag{3.17}$$

If $N \ge 7$, by the mean value theorem ($\theta \in (0,1)$) we get

$$\begin{aligned}
&\Big\| f(P_\varepsilon U_{\lambda_i,\xi_i/\varepsilon^\alpha} + \phi_i^2) - f(P_\varepsilon U_{\lambda_i,\xi_i/\varepsilon^\alpha} + \phi_i^1) \\
&\qquad - f'(P_\varepsilon U_{\lambda_i,\xi_i/\varepsilon^\alpha})(\phi_i^2 - \phi_i^1)\Big\|_{\frac{2N}{N+2}} \\
&\quad = \Big\| [f'(P_\varepsilon U_{\lambda_i,\xi_i/\varepsilon^\alpha} + \phi_i^2 + \theta(\phi_i^1 - \phi_i^2)) \\
&\qquad - f'(P_\varepsilon U_{\lambda_i,\xi_i/\varepsilon^\alpha})](\phi_i^2 - \phi_i^1)\Big\|_{\frac{2N}{N+2}} \\
&\quad \le c\left(\|\phi_i^1 - \phi_i^2\|^p_{\frac{2N}{N-2}} + \|\phi_i^2\|^{p-1}_{\frac{2N}{N-2}} \|\phi_i^1 - \phi_i^2\|_{\frac{2N}{N-2}} \right).
\end{aligned} \tag{3.18}$$

Moreover

$$\begin{aligned}
&\|G(\varepsilon^\alpha y,\, P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha} + \phi_1^2,\, P_\varepsilon U_{\lambda_2,\xi_2^1/\varepsilon^\alpha} + \phi_2^2) \\
&\qquad - G(\varepsilon^\alpha y,\, P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha} + \phi_1^1,\, P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha} + \phi_2^1)\|_{\frac{2N}{N+2}} \\
&\quad \le \varepsilon^{-2\alpha} \max\{\|a\|_\infty, \|b\|_\infty, \|c\|_\infty, \|d\|_\infty\} \|\phi^2 - \phi^1\|_{\frac{2N}{N-2}}.
\end{aligned} \tag{3.19}$$

By (3.17)-(3.19) we get the claim.
If $N = 5, 6$ we proceed in a similar way. □

4 The reduced problem

In this section we are finding (λ, ξ) such that also equation (2.9) is verified, namely for $j, l = 0, 1, \ldots, N$

$$\begin{aligned}
0 = \Big(&\big(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha} + \phi_{1_{\varepsilon,\lambda,\xi}},\, P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha} + \phi_{2_{\varepsilon,\lambda,\xi}}\big) \\
&- \mathcal{I}^*_\varepsilon \big[F(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha} + \phi_{1_{\varepsilon,\lambda,\xi}},\, P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha} + \phi_{2_{\varepsilon,\lambda,\xi}}) \\
&+ \varepsilon^{2\alpha+1} G(\varepsilon^\alpha y,\, P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha} + \phi_{1_{\varepsilon,\lambda,\xi}},\, P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha} + \phi_{2_{\varepsilon,\lambda,\xi}})\big], \\
&(P_\varepsilon \psi^j_{\lambda_1,\xi_1/\varepsilon^\alpha},\, P_\varepsilon \psi^l_{\lambda_2,\xi_2/\varepsilon^\alpha})\Big)_{\mathrm{H}}.
\end{aligned} \tag{4.20}$$

We need the expansion of the R.H.S. of (4.20).

First of all, arguing as in Proposition 2.1 of [14], we get the following result.

Lemma 4.1. *For $i = 1, 2$ and $j = 1, \dots, N$, it holds*

$$\Big(P_\varepsilon U_{\lambda_i,\xi_i/\varepsilon^\alpha} + \phi_{i\,\varepsilon,\lambda,\xi} - i^*_\varepsilon[f(P_\varepsilon U_{\lambda_i,\xi_i/\varepsilon^\alpha} + \phi_{i\,\varepsilon,\lambda,\xi})],\ P_\varepsilon \psi^j_{\lambda_i,\xi_i/\varepsilon^\alpha}\Big)_{H^1_0}$$
$$= -\frac{A^2}{2}\frac{\partial}{\partial \xi_i^j}[\tau(\xi_i)\lambda_i^{N-2}]\varepsilon^{\alpha(N-1)} + O(\|\phi_{\varepsilon,\lambda,\xi}\|^2) + \varepsilon^{\alpha\frac{N}{2}} O(\|\phi_{\varepsilon,\lambda,\xi}\|)$$

and

$$\big(P_\varepsilon U_{\lambda_i,\xi_i/\varepsilon^\alpha} + \phi_{i\,\varepsilon,\lambda,\xi} - i^*_\varepsilon[f(P_\varepsilon U_{\lambda_i,\xi_i/\varepsilon^\alpha} + \phi_{i\,\varepsilon,\lambda,\xi})],\ P_\varepsilon \psi^0_{\lambda_i,\xi_i/\varepsilon^\alpha}\big)$$
$$= \frac{A^2}{2}\frac{\partial}{\partial \lambda_i}[\tau(\xi_i)\lambda_i^{N-2}]\varepsilon^{\alpha(N-2)} + O(\|\phi_{\varepsilon,\lambda,\xi}\|^2) + \varepsilon^{\alpha\frac{N-2}{2}} O(\|\phi_{\varepsilon,\lambda,\xi}\|)$$

as ε goes to zero, uniformly with respect to $(\lambda, \xi) \in \mathcal{O}_\mu$. Here $A = \int_{\mathbb{R}^N} U^p_{1,0}(z)dz$.

Secondly we have the following expansion.

Lemma 4.2. *Assume that $w(x) \in C^1(\bar{\Omega})$, then, for $j = 1, \dots, N$,*

$$\int_{\Omega_\varepsilon} w(\varepsilon^\alpha y) P_\varepsilon U_{\lambda_i,\xi_i/\varepsilon^\alpha}(y) P_\varepsilon \psi^j_{\lambda_i,\xi_i/\varepsilon^\alpha}(y)\, dy = -\frac{B}{2}\frac{\partial}{\partial \xi_i^j}\left[w(\xi_i)\lambda_i^2\right]\varepsilon^\alpha\,(1 + o(1))$$

and

$$\int_{\Omega_\varepsilon} w(\varepsilon^\alpha y) P_\varepsilon U_{\lambda_i,\xi_i/\varepsilon^\alpha}(y) P_\varepsilon \psi^0_{\lambda_i,\xi_i/\varepsilon^\alpha}(y)\, dy = \frac{B}{2}\frac{\partial}{\partial \lambda_i}\left[w(\xi_i)\lambda_i^2\right](1 + o(1)),$$

as ε goes to zero, uniformly with respect to $(\lambda, \xi) \in \mathcal{O}_\mu$. Here $B = \int_{\mathbb{R}^N} U^2_{1,0}(z)dz$.

Moreover if $i \neq h$ and $j = 1, \dots, N$

$$\int_{\Omega_\varepsilon} w(\varepsilon^\alpha y) P_\varepsilon U_{\lambda_h,\xi_h/\varepsilon^\alpha}(y) P_\varepsilon \psi^j_{\lambda_i,\xi_i/\varepsilon^\alpha}(y)\, dy = O\left(\varepsilon^{\alpha(N-3)}\right) \tag{4.21}$$

and

$$\int_{\Omega_\varepsilon} w(\varepsilon^\alpha y) P_\varepsilon U_{\lambda_h,\xi_h/\varepsilon^\alpha}(y) P_\varepsilon \psi^0_{\lambda_i,\xi_i/\varepsilon^\alpha}(y)\, dy = O\left(\varepsilon^{\alpha(N-4)}\right), \tag{4.22}$$

as ε goes to zero, uniformly with respect to $(\lambda, \xi) \in \mathcal{O}_\mu$.

Proof. For $j = 0, 1, \dots, N$, let us compute

$$\begin{aligned}\int_{\Omega_\varepsilon} w(\varepsilon^\alpha y) P_\varepsilon U_{\lambda_i,\xi_i/\varepsilon^\alpha} P_\varepsilon \psi^j_{\lambda_i,\xi_i/\varepsilon^\alpha} &= \int_{\Omega_\varepsilon} w(\varepsilon^\alpha y) U_{\lambda_i,\xi_i/\varepsilon^\alpha} \psi^j_{\lambda_i,\xi_i/\varepsilon^\alpha} \\ &+ \int_{\Omega_\varepsilon} w(\varepsilon^\alpha y)\left(P_\varepsilon U_{\lambda_i,\xi_i/\varepsilon^\alpha} - U_{\lambda_i,\xi_i/\varepsilon^\alpha}\right)\psi^j_{\lambda_i,\xi_i/\varepsilon^\alpha} \\ &+ \int_{\Omega_\varepsilon} w(\varepsilon^\alpha y) P_\varepsilon U_{\lambda_i,\xi_i/\varepsilon^\alpha}\left(P_\varepsilon \psi^j_{\lambda_i,\xi_i/\varepsilon^\alpha} - \psi^j_{\lambda_i,\xi_i/\varepsilon^\alpha}\right).\end{aligned} \tag{4.23}$$

Furthermore

$$\begin{aligned}&\int_{\Omega_\varepsilon} w(\varepsilon^\alpha y) U_{\lambda_i,\xi_i/\varepsilon^\alpha} \psi^j_{\lambda_i,\xi_i/\varepsilon^\alpha} dy \\ &\quad = \varepsilon^{-\alpha N} \int_{\Omega} w(x) U_{\lambda_i,\xi_i/\varepsilon^\alpha}\left(\frac{x}{\varepsilon^\alpha}\right) \psi^j_{\lambda_i,\xi_i/\varepsilon^\alpha}\left(\frac{x}{\varepsilon^\alpha}\right) dx.\end{aligned} \tag{4.24}$$

For $j = 1, \dots, N$, (4.24) is equal to

$$\begin{aligned}&\lambda_i \int_{\frac{\Omega-\xi_i}{\lambda_i\varepsilon^\alpha}} w(\xi_i + \lambda_i\varepsilon^\alpha z) U_{1,0} \left.\frac{\partial U_{1,\xi_i}}{\partial \xi_i^j}\right|_{\xi=0} dz \\ &= \frac{1}{2}\lambda_i \int_{\frac{\Omega-\xi_i}{\lambda_i\varepsilon^\alpha}} w(\xi_i + \lambda_i\varepsilon^\alpha z) \frac{\partial U^2_{1,0}}{\partial z^j} dz \\ &= \frac{1}{2}\lambda_i \left[-\int_{\frac{\Omega-\xi_i}{\lambda_i\varepsilon^\alpha}} \frac{\partial w}{\partial z^j}(\xi_i + \lambda_i\varepsilon^\alpha z) U^2_{1,0}(z) dz \right. \\ &\quad \left. + \int_{\partial\left(\frac{\Omega-\xi_i}{\lambda_i\varepsilon^\alpha}\right)} w(\xi_i + \lambda_i\varepsilon^\alpha z) U^2_{1,0}(z) d\sigma \right] \\ &= -\frac{1}{2}\lambda_i\varepsilon^\alpha\lambda_i \int_{\frac{\Omega-\xi_i}{\lambda_i\varepsilon^\alpha}} \frac{\partial w}{\partial \xi^j}(\xi_i + \lambda_i\varepsilon^\alpha z) U^2_{1,0}(z) dz + o(\varepsilon^\alpha) \\ &= -\lambda_i^2\varepsilon^\alpha \frac{B}{2} \frac{\partial w(\xi_i)}{\partial \xi_i^j}(1 + o(1)).\end{aligned} \tag{4.25}$$

For $j = 0$, (4.24) is equal to

$$\lambda_i w(\xi_i) \int_{\mathbb{R}^N} U_{1,0} \left.\frac{\partial U_{\lambda_i,0}}{\partial \lambda_i}\right|_{\lambda_i=1} (1 + o(1)) = B\lambda_i w(\xi_i)(1 + o(1)). \tag{4.26}$$

Now, by Lemma 5.4 in [13]

$$\begin{aligned}
&\left|\int_{\Omega_\varepsilon} w(\varepsilon^\alpha y)\left(P_\varepsilon U_{\lambda_i,\xi_i/\varepsilon^\alpha} - U_{\lambda_i,\xi_i/\varepsilon^\alpha}\right)\psi^j_{\lambda_i,\xi_i/\varepsilon^\alpha}\right| \\
&\qquad \le \|w\|_\infty \int_{\Omega_\varepsilon}\left|\left(P_\varepsilon U_{\lambda_i,\xi_i/\varepsilon^\alpha} - U_{\lambda_i,\xi_i/\varepsilon^\alpha}\right)\psi^j_{\lambda_i,\xi_i/\varepsilon^\alpha}\right| \\
&\qquad \le \begin{cases} o(\varepsilon^\alpha) & \text{if } j = 1,\dots,N \\ o(1) & \text{if } j = 0. \end{cases}
\end{aligned} \tag{4.27}$$

and by Lemmas 5.3 and 5.5 in [13] it follows that

$$\begin{aligned}
&\left|\int_{\Omega_\varepsilon} w(\varepsilon^\alpha y) P_\varepsilon U_{\lambda_i,\xi_i/\varepsilon^\alpha}\left(P_\varepsilon \psi^j_{\lambda_i,\xi_i/\varepsilon^\alpha} - \psi^j_{\lambda_i,\xi_i/\varepsilon^\alpha}\right)\right| \\
&\qquad \le c\|w\|_\infty \|P_\varepsilon \psi^j_{\lambda_i,\xi_i/\varepsilon^\alpha} - \psi^j_{\lambda_i,\xi_i/\varepsilon^\alpha}\|_{\frac{2N}{N-2}} \|P_\varepsilon U_{\lambda_i,\xi_i/\varepsilon^\alpha}\|_{\frac{2N}{N+2}} \\
&\qquad \le \begin{cases} o(\varepsilon^\alpha) & \text{if } j = 1,\dots,N \\ o(1) & \text{if } j = 0. \end{cases}
\end{aligned} \tag{4.28}$$

For $i \neq h$ and $j = 1,\dots,N$ we have

$$\begin{aligned}
&\left|\int_{\Omega_\varepsilon} w(\varepsilon^\alpha y) P_\varepsilon U_{\lambda_h,\xi_h/\varepsilon^\alpha}(y) P_\varepsilon \psi^j_{\lambda_i,\xi_i/\varepsilon^\alpha}(y)\,dy\right| \\
&\le \|w\|_\infty \varepsilon^{-\alpha N}\int_\Omega U_{\lambda_h,\xi_h/\varepsilon^\alpha}\left(\frac{x}{\varepsilon^\alpha}\right)\left(\left|\psi^j_{\lambda_i,\xi_i/\varepsilon^\alpha}\left(\frac{x}{\varepsilon^\alpha}\right)\right| + \max_{\partial\Omega}\left|\psi^j_{\lambda_i,\xi_i/\varepsilon^\alpha}\left(\frac{x}{\varepsilon^\alpha}\right)\right|\right)dx \\
&= O\left(\varepsilon^{\alpha(N-3)}\right).
\end{aligned} \tag{4.29}$$

A similar argument proves (4.22). □

Finally we can give the expansion of (4.20).

Proposition 4.3. *It holds*

$$\begin{aligned}
&\Big(\big(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha} + \phi_{1\varepsilon,\lambda,\xi},\, P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha} + \phi_{2\varepsilon,\lambda,\xi}\big) \\
&\quad -\mathcal{I}^*_\varepsilon\big[F(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha} + \phi_{1\varepsilon,\lambda,\xi},\, P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha} + \phi_{2\varepsilon,\lambda,\xi}) \\
&\quad +\varepsilon^{2\alpha+1} G(\varepsilon^\alpha y,\, P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha} + \phi_{1\varepsilon,\lambda,\xi},\, P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha} + \phi_{2\varepsilon,\lambda,\xi})\big], \\
&\quad \left(P_\varepsilon \psi^j_{\lambda_1,\xi_1/\varepsilon^\alpha}, 0\right)\Big)_H \\
&\quad = \begin{cases} \varepsilon^{\frac{N-1}{N-4}}\left[-\dfrac{\partial\Psi}{\partial\xi_1^j}(\lambda,\xi) + o(1)\right] & \textit{if } j = 1,\dots,N, \\ \varepsilon^{\frac{N-2}{N-4}}\left[\dfrac{\partial\Psi}{\partial\lambda_1}(\lambda,\xi) + o(1)\right] & \textit{if } j = 0, \end{cases}
\end{aligned} \tag{4.30}$$

and

$$\begin{aligned}&\Big(\big(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha}+\phi_{1\varepsilon,\lambda,\xi},\,P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha}+\phi_{2\varepsilon,\lambda,\xi}\big)\\&\quad-\mathcal{I}^*_\varepsilon\big[F(P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha}+\phi_{1\varepsilon,\lambda,\xi},\,P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha}+\phi_{2\varepsilon,\lambda,\xi})\\&\quad+\varepsilon^{2\alpha+1}G(\varepsilon^\alpha y,\,P_\varepsilon U_{\lambda_1,\xi_1/\varepsilon^\alpha}+\phi_{1\varepsilon,\lambda,\xi},\,P_\varepsilon U_{\lambda_2,\xi_2/\varepsilon^\alpha}+\phi_{2\varepsilon,\lambda,\xi})\big],\\&\quad\big(0,\,P_\varepsilon\psi^l_{\lambda_2,\xi_2/\varepsilon^\alpha}\big)\Big)_H\\&\quad=\begin{cases}\varepsilon^{\frac{N-1}{N-4}}\Big[-\frac{\partial\Psi}{\partial\xi_2^l}(\lambda,\xi)+o(1)\Big] & \text{if } l=1,\dots,N,\\ \varepsilon^{\frac{N-2}{N-4}}\Big[\frac{\partial\Psi}{\partial\lambda_2}(\lambda,\xi)+o(1)\Big] & \text{if } l=0,\end{cases}\end{aligned}\tag{4.31}$$

as ε *goes to zero, uniformly with respect to* $(\lambda,\xi)\in\mathcal{O}_\mu$, *where* $\Psi:\big(\mathbb{R}^+\big)^2\times\Omega^2\longrightarrow\mathbb{R}$ *is defined by (see Lemma* 4.1 *and Lemma* 4.2*).*

$$\Psi(\lambda,\xi)=\frac{A^2}{2}\Big[\tau(\xi_1)\lambda_1^{N-2}+\tau(\xi_2)\lambda_2^{N-2}\Big]-\frac{B}{2}\Big[a(\xi_1)\lambda_1^2+d(\xi_2)\lambda_2^2\Big].\tag{4.32}$$

Proof. The claim easily follows by Lemma 4.1, Lemma 4.2 and (3.11). We point out that α is chosen so that $\alpha(N-1)=3\alpha+1$. □

Now we get the following necessary condition.

Theorem 4.4. *Let*

$$(u_{1\varepsilon},u_{2\varepsilon})=(P_\varepsilon U_{\lambda_{1\varepsilon},\xi_{1\varepsilon}/\varepsilon^\alpha}+\phi_{1\varepsilon,\lambda_{1\varepsilon},\xi_{1\varepsilon}},\,P_\varepsilon U_{\lambda_{2\varepsilon},\xi_{2\varepsilon}/\varepsilon^\alpha}+\phi_{2\varepsilon,\lambda_{2\varepsilon},\xi_{2\varepsilon}})$$

be a family of solutions of (2.6) *(see Proposition* 3.2*) such that* $\lim\limits_{\varepsilon\to0}\lambda_{i_\varepsilon}=\lambda_i>0$ *and* $\lim\limits_{\varepsilon\to0}\xi_{i\,\varepsilon}=\xi_i\in\Omega$, $i=1,2$, *with* $\xi_1\neq\xi_2$. *Then* (λ,ξ) *is a critical point of the function* Ψ.

Proof. By Proposition 4.3 it follows $\nabla\Psi(\lambda_\varepsilon,\xi_\varepsilon)+o(1)=0$, uniformly with respect to ε. Then passing to the limit we get the claim. □

Conversely we prove the following sufficient condition.

Theorem 4.5. *Let* (λ,ξ) *be a stable critical point of the function* Ψ *(see Definition* 1.2*) with* $\lambda_1,\lambda_2>0$ *and* $\xi_1\neq\xi_2$. *Then there exists a family of solutions*

$$(u_{1\varepsilon},u_{2\varepsilon})=(P_\varepsilon U_{\lambda_{1\varepsilon},\xi_{1\varepsilon}/\varepsilon^\alpha}+\phi_{1\varepsilon,\lambda_{1\varepsilon},\xi_{1\varepsilon}},\,P_\varepsilon U_{\lambda_{2\varepsilon},\xi_{2\varepsilon}/\varepsilon^\alpha}+\phi_{2\varepsilon,\lambda_{2\varepsilon},\xi_{2\varepsilon}})$$

of problem (2.6). *Moreover* $\lim\limits_{\varepsilon\to0}\lambda_\varepsilon=\lambda^*$ *and* $\lim\limits_{\varepsilon\to0}\xi_\varepsilon=\xi^*$, *where* (λ^*,ξ^*) *is a critical point of* Ψ *with* $\Psi(\lambda^*,\xi^*)=\Psi(\lambda,\xi)$.

Proof. By Proposition 4.3 and Definition 1.2 we get that for ε small enough there exists $(\lambda_\varepsilon, \xi_\varepsilon)$ in a neighbourhood V of (λ_0, ξ_0) such that $\nabla\Psi(\lambda_\varepsilon, \xi_\varepsilon) + o(1) = 0$. Up to a subsequence we can assume that $\lambda_\varepsilon \to \lambda^*$ and $\xi_\varepsilon \to \xi^*$. Therefore (λ^*, ξ^*) is a critical point of the function Ψ and by Definition 1.2 it follows that $\psi(\lambda^*, \xi^*) = \psi(\lambda, \xi)$. □

We remark that if (λ, ξ) is an isolated critical point of Ψ then $(\lambda^*, \xi^*) = (\lambda, \xi)$ (see Definition 1.2).

We also point out that the maximum principle easily implies the following result.

Proposition 4.6. *If we assume $b(x), c(x) \geq 0$, $x \in \Omega$, then the solutions $(u_{1_\varepsilon}, u_{2_\varepsilon})$ of system* (1.1) *given in Theorem* 4.5 *are positive in Ω.*

We are in position to prove our first main result.

Proof of Theorem 1.3. First notice that, in general, if $f : \Omega \to \mathbb{R}$ is a smooth function, $\alpha \in \mathbb{R} \setminus \{0\}$ and $\bar{x} \in \Omega$ is such that $f(\bar{x}) \neq 0$, then $\bar{x}$ is a stable critical point for f if and only if $\bar{x}$ is a stable critical point for $|f|^\alpha$. Then our claim follows from Theorem 4.5 and Lemma 5.7 in [13], taking $\delta_{i\varepsilon} = \lambda_{i\varepsilon}\varepsilon^{\frac{1}{N-4}}$. □

Example 4.7. Let Ω be a "dumb-bell with thin handle" (see [14], Section 3) and let $a(x) = d(x) = 1$ for any $x \in \Omega$. Then there exist two families of solutions to (1.1) which blow-up and concentrate at two different points of Ω.

Example 4.8. Let a and d be positive functions with disjoint supports. Then there exists a family of solutions to (1.1) which blow-up and concentrate at two different points of Ω.

5 The symmetric case

In this section we assume that Ω is a symmetric domain and that a, b, c, d are symmetric functions (see (1.3), (1.4)). Let $H_s = \{(u, v) \in \mathrm{H} : u, v \text{ are symmetric}\}$.

We are looking for a solution of (2.6) in the space H_s as $(u_1(x), u_2(x)) = \left(P_\varepsilon U_{\lambda_1,0}(x) + \phi_{1_\varepsilon}(x), P_\varepsilon U_{\lambda_2,0}(x) + \phi_{2_\varepsilon}(x)\right)$, where the rest term $\phi_\varepsilon(x) = \left(\phi_{1_\varepsilon}(x), \phi_{2_\varepsilon}(x)\right)$ belongs to the space $K^\perp_{\varepsilon,\xi,0} \cap \mathrm{H}_s$.

Arguing exactly as in the previous Sections 2 and 3 we can prove that

Proposition 5.1. *For any $\mu \in (0, 1)$ there exist $R, \varepsilon_0 > 0$ such that for every $\lambda \in (\mu, 1/\mu)$ and for any $\varepsilon \in (0, \varepsilon_0)$ there exists a unique $\phi_{\varepsilon,\lambda} = \left(\phi_{1_{\varepsilon,\lambda}}, \phi_{2_{\varepsilon,\lambda}}\right) \in$*

$K^{\perp}_{\varepsilon,\xi,0} \cap H_s$ *such that*

$$\begin{aligned}\Pi^{\perp}_{\varepsilon,\lambda,0}\Big\{&\big(P_\varepsilon U_{\lambda_1,0}+\phi_{1_{\varepsilon,\lambda}},\,P_\varepsilon U_{\lambda_2,0}+\phi_{2_{\varepsilon,\lambda}}\big)\\&-\mathfrak{I}^*_\varepsilon\Big[F\big(P_\varepsilon U_{\lambda_1,0}+\phi_{1_{\varepsilon,\lambda}},\,P_\varepsilon U_{\lambda_2,0}+\phi_{2_{\varepsilon,\lambda}}\big)\\&+\varepsilon^{2\alpha+1}G\big(\varepsilon^\alpha y,\,P_\varepsilon U_{\lambda_1,0}+\phi_{1_{\varepsilon,\lambda}},\,P_\varepsilon U_{\lambda_2,0}+\phi_{2_{\varepsilon,\lambda}}\big)\Big]\Big\}=0.\end{aligned}\tag{5.33}$$

Moreover $\|\phi_{\varepsilon,\lambda}\|$ *satisfies estimate* (3.11).

The problem reduces to find parameters λ_1 and λ_2 such that also Equation (2.9) is verified, namely

$$\begin{aligned}0=\Big(&\big(P_\varepsilon U_{\lambda_1,0}+\phi_{1\varepsilon,\lambda},\,P_\varepsilon U_{\lambda_2,0}+\phi_{2\varepsilon,\lambda}\big)\\&-\mathfrak{I}^*_\varepsilon\left[F(P_\varepsilon U_{\lambda_1,0}+\phi_{1\varepsilon,\lambda},\,P_\varepsilon U_{\lambda_2,0}+\phi_{2\varepsilon,\lambda})\right.\\&\left.+\varepsilon^{2\alpha+1}G(\varepsilon^\alpha y,\,P_\varepsilon U_{\lambda_1,0}+\phi_{1\varepsilon,\lambda},\,P_\varepsilon U_{\lambda_2,0}+\phi_{2\varepsilon,\lambda})\right],\\&\big(sP_\varepsilon\psi^0_{\lambda_1,0},\,tP_\varepsilon\psi^0_{\lambda_2,0}\big)\Big)_{\mathrm{H}}\qquad\forall\,(s,t)\in\mathbb{R}^2.\end{aligned}\tag{5.34}$$

Arguing as in Lemma 4.2, we can prove that:

Lemma 5.2. *It holds if* $i\neq j$

$$\int_{\Omega_\varepsilon} w(\varepsilon^\alpha y)P_\varepsilon U_{\lambda_i,0}(y)P_\varepsilon\psi^0_{\lambda_j,0}(y)\,dy = w(0)\frac{\partial}{\partial\lambda_j}\left[\int_{\mathbb{R}^N}U_{\lambda_i,0}(y)U_{\lambda_j,0}(y)dy\right](1+o(1))$$

and if $i=j$

$$\int_{\Omega_\varepsilon} w(\varepsilon^\alpha y)P_\varepsilon U_{\lambda_i,0}(y)P_\varepsilon\psi^0_{\lambda_i,0}(y)\,dy = \frac{1}{2}w(0)\frac{\partial}{\partial\lambda_j}\left[\int_{\mathbb{R}^N}U^2_{\lambda_i,0}(y)dy\right](1+o(1)),$$

as ε *goes to zero, uniformly with respect to* $\lambda\in[\mu,1/\mu]$.

Arguing as in the proof of Proposition 4.3, we get the expansion of the R.H.S. of (5.34).

Proposition 5.3. *It holds*

$$\begin{aligned}&\Big(\big(P_\varepsilon U_{\lambda_1,0}+\phi_{1\varepsilon,\lambda},\, P_\varepsilon U_{\lambda_2,0}+\phi_{2\varepsilon,\lambda}\big)\\ &\quad - \mathfrak{I}^*_\varepsilon\big[F(P_\varepsilon U_{\lambda_1,0}+\phi_{1\varepsilon,\lambda},\, P_\varepsilon U_{\lambda_2,0}+\phi_{2\varepsilon,\lambda})\\ &\quad + \varepsilon^{2\alpha+1}G(\varepsilon^\alpha y,\, P_\varepsilon U_{\lambda_1,0}+\phi_{1\varepsilon,\lambda},\, P_\varepsilon U_{\lambda_2,0}+\phi_{2\varepsilon,\lambda})\big],\, \big(P_\varepsilon \psi^0_{\lambda_1,0},\, 0\big)\Big)_H \\ &= \varepsilon^{\frac{N-2}{N-4}}\big[\sigma_1(\lambda)+o(1)\big]\end{aligned} \tag{5.35}$$

and

$$\begin{aligned}&\Big(\big(P_\varepsilon U_{\lambda_1,0}+\phi_{1\varepsilon,\lambda},\, P_\varepsilon U_{\lambda_2,0}+\phi_{2\varepsilon,\lambda}\big)\\ &\quad - \mathfrak{I}^*_\varepsilon\big[F(P_\varepsilon U_{\lambda_1,0}+\phi_{1\varepsilon,\lambda},\, P_\varepsilon U_{\lambda_2,0}+\phi_{2\varepsilon,\lambda})\\ &\quad + \varepsilon^{2\alpha+1}G(\varepsilon^\alpha y,\, P_\varepsilon U_{\lambda_1,0}+\phi_{1\varepsilon,\lambda},\, P_\varepsilon U_{\lambda_2,0}+\phi_{2\varepsilon,\lambda})\big],\, \big(0,\, P_\varepsilon \psi^0_{\lambda_2,0}\big)\Big)_H \\ &= \varepsilon^{\frac{N-2}{N-4}}\,[\sigma_2(\lambda)+o(1)]\end{aligned} \tag{5.36}$$

as ε *goes to zero, uniformly with respect to* $\lambda \in [\mu, 1/\mu]$. *Here the function* $\sigma : (0,+\infty)\times(0,+\infty)\to\mathbb{R}^2$, *is defined by*

$$\begin{aligned}\sigma_1(\lambda) &= \frac{\partial}{\partial\lambda_1}\left[\frac{1}{2}A^2\tau(0)\lambda_1^{N-2} - \frac{1}{2}a(0)\int_{\mathbb{R}^N} U^2_{\lambda_1} - b(0)\int_{\mathbb{R}^N} U_{\lambda_1}U_{\lambda_2}\right]\\ \sigma_2(\lambda) &= \frac{\partial}{\partial\lambda_2}\left[\frac{1}{2}A^2\tau(0)\lambda_2^{N-2} - \frac{1}{2}d(0)\int_{\mathbb{R}^N} U^2_{\lambda_2} - c(0)\int_{\mathbb{R}^N} U_{\lambda_1}U_{\lambda_2}\right].\end{aligned} \tag{5.37}$$

Arguing as in the proof of Theorem 4.4, we have the following necessary condition.

Theorem 5.4. *Let* $(u_{1\varepsilon}, u_{2\varepsilon}) = (P_\varepsilon U_{\lambda_{1\varepsilon},0}+\phi_{1\varepsilon,\lambda_{1\varepsilon}},\, P_\varepsilon U_{\lambda_{2\varepsilon},0}+\phi_{2\varepsilon,\lambda_{2\varepsilon}})$ *(see Proposition* 5.1*) be a family of solutions of* (2.6) *such that* $\lim_{\varepsilon\to 0}\lambda_{i_\varepsilon} = \lambda_i > 0$. *Then* $\sigma(\lambda)=0$ *(see* (5.37)*).*

Conversely, given the following notion of stable zero for a vector field, we can prove a sufficient condition.

Definition 5.5. *Let $G : \Omega \to \mathbb{R}^N$ be a C^1-function, we say that ξ_0 is a stable zero of G if $G(\xi_0) = 0$ and there exists a neighbourhood $V \subset\subset \Omega$ of ξ_0 such that $G(\xi) \neq 0 \quad \forall\, \xi \in \partial V$ and* $\deg\big(G, \bar{V}, 0\big) \neq 0$.

Theorem 5.6. *If $\lambda^* = (\lambda_1^*, \lambda_2^*)$ is a stable zero of the function σ (see (5.37)), then there exists a family of symmetric solutions $(u_{1\varepsilon}, u_{2\varepsilon}) = (P_\varepsilon U_{\lambda_{1\varepsilon},0} + \phi_{1\varepsilon,\lambda_{1\varepsilon}}, P_\varepsilon U_{\lambda_{2\varepsilon},0} + \phi_{2\varepsilon,\lambda_{2\varepsilon}})$ of problem (2.6) with $\lim_{\varepsilon\to 0} \lambda_\varepsilon = \lambda^*$.*

Proof. By Proposition 5.3 it follows that

$$
\begin{aligned}
\Pi_{\varepsilon,\lambda,0}\Big\{&\big(P_\varepsilon U_{\lambda_1,0} + \phi_{1\varepsilon,\lambda},\, P_\varepsilon U_{\lambda_2,0} + \phi_{2\varepsilon,\lambda}\big)\\
&- \mathcal{I}_\varepsilon^*\big[F(P_\varepsilon U_{\lambda_1,0} + \phi_{1\varepsilon,\lambda},\, P_\varepsilon U_{\lambda_2,0} + \phi_{2\varepsilon,\lambda})\\
&+\varepsilon^{2\alpha+1} G(\varepsilon^\alpha y,\, P_\varepsilon U_{\lambda_1,0} + \phi_{1\varepsilon,\lambda},\, P_\varepsilon U_{\lambda_2,0} + \phi_{2\varepsilon,\lambda})\big]\Big\}\\
= {}&\varepsilon^{\frac{N-2}{N-4}}\left[\sigma(\lambda) + o(1)\right].
\end{aligned}
$$

By Definition 5.5 we get that for ε small enough there exists λ_ε in a neighbourhood of λ^* such that $\sigma(\lambda_\varepsilon) + o(1) = 0$ and the claim follows. □

In order to find stable zeroes of the function σ, we need the following technical lemma.

Lemma 5.7. *Let $\phi : (0, +\infty) \to \mathbb{R}$ be defined by*

$$\phi(t) := \int_{\mathbb{R}^N} \psi_{1,0}^0(y) U_{t,0}(y)\, dy. \tag{5.38}$$

Let $N \geq 5$. Then

(i) $\lim_{t\to 0} t^{-\frac{N-2}{2}}\phi(t) = C$, *where* $C := C_N^2 \frac{N-2}{2} \int_{\mathbb{R}^N} \frac{|y|^2-1}{(|y|^2+1)^{\frac{N}{2}}} \frac{1}{|y|^{N-2}} dy$, $C < 0$ *if* $N \geq 7$, $C = 0$ *if* $N = 6$, $C > 0$ *if* $N = 5$;

(ii) $\lim_{t\to +\infty} t^{\frac{N-6}{2}}\phi(t) > 0$;

(iii) $\phi(1) > 0$ *and* $\phi'(1) > 0$.

Moreover if $N \geq 7$

(iv) *there exists $m \in (0, 1)$ such that ϕ is increasing in $(m, 1/m)$ and is decreasing in $(0, m) \cup (1/m, +\infty)$;*

(v) *there exists a unique* $\zeta \in (0,1)$ *such that* $\phi(\zeta) = 0$

and if $N = 5, 6$

(vi) $\phi'(t) > 0$ *for any* $t > 0$.

Proof. Since

$$\phi(t) = C_N^2 \frac{N-2}{2} t^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{|y|^2 - 1}{(|y|^2+1)^{\frac{N}{2}}} \frac{1}{(|y|^2+t^2)^{\frac{N-2}{2}}} dy, \tag{5.39}$$

(i) follows. Moreover, setting $y = tx$ in (5.39), we have

$$t^{\frac{N-6}{2}} \phi(t) \longrightarrow C_N^2 \frac{N-2}{2} \int_{\mathbb{R}^N} \frac{1}{(|x|^2+1)^{\frac{N-2}{2}}} \frac{1}{|x|^{N-2}} dy \quad \text{as } t \to +\infty,$$

that proves (ii).

By (5.39), $\phi(1) > 1$ and

$$\phi'(t) = \left(C_N \frac{N-2}{2} \right)^2 t^{\frac{N-4}{2}} \int_{\mathbb{R}^N} \frac{|y|^2 - 1}{(|y|^2+1)^{\frac{N}{2}}} \frac{|y|^2 - t^2}{(|y|^2+t^2)^{\frac{N}{2}}} dy, \tag{5.40}$$

from which (iii) follows.

Now let us prove (iv) and (vi). First we point out that, if we make the change of variable $y = tx$ in (5.40), we deduce that $\phi'(t) = \phi'\left(\frac{1}{t}\right)$ for any $t > 0$. So it is enough to study ϕ in $(1, +\infty)$ and to show that there exists $M \in (1, +\infty)$ such that ϕ is increasing in $(1, M)$ and is decreasing in $(M, +\infty)$.

Using polar co-ordinates we can rewrite (5.40) as

$$\phi'(t) = \alpha_1 t^{\frac{N-4}{2}} \int_0^\infty \rho^{N-1} \frac{\rho^2 - 1}{(\rho^2+1)^{\frac{N}{2}}} \frac{\rho^2 - t^2}{(\rho^2+t^2)^{\frac{N}{2}}} d\rho, \tag{5.41}$$

where $\alpha_1 = \left(C_N \frac{N-2}{2} \right)^2 \operatorname{meas}(S^{N-1})$ and S^{N-1} is the $(N-1)-$dimensional unit sphere.

Using hypergeometric functions (see [11], Section 9.1) and their properties (in particular, 9.137 in [11]) we can compute

$$\begin{aligned} \phi'(t) = \alpha_2 t^{-\frac{N+4}{2}} \Bigg[& -N(N-6) F\left(\frac{N}{2}, \frac{N+2}{2}, N, \frac{t^2-1}{t^2} \right)(t^2+1) \\ & +2(N-2)^2 F\left(\frac{N}{2}, \frac{N}{2}, N, \frac{t^2-1}{t^2} \right) t^2 \Bigg], \end{aligned} \tag{5.42}$$

where $\alpha_2 = \alpha_1 \frac{\Gamma(\frac{N-4}{2})\Gamma(\frac{N}{2})}{8\Gamma(N)}$. If $N = 5, 6$, then (5.42) directly imply (vi).

If $N \geq 7$, let us evaluate the second derivative $\phi''(t)$ at any critical point t of ϕ, namely $\phi'(t) = 0$:

$$\phi''(t) = -\alpha_3 \frac{t^{-\frac{N+6}{2}}}{t^4 - 1} F\left(\frac{N}{2}, \frac{N}{2}, N, \frac{t^2-1}{t^2}\right)\left(t^4 - 2\frac{N+2}{N-6}t^2 + 1\right), \quad (5.43)$$

where $\alpha_3 = \alpha_2 \frac{N+2}{2} \frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N+2}{2}\right)}{\Gamma(N+1)}$.

Now, since (ii) and (iii) hold, there exists a local maximum point $M \in (1, +\infty)$ such that $\phi'(t) \geq 0$ for any $t \in (1, M)$. By (5.42) we deduce that $M^2 \geq \frac{N+2+4\sqrt{N-2}}{N-6}$. If M is not the global maximum point of ϕ in $(1, +\infty)$ then there exists a local minimum point t of ϕ with $t > M$. Then t must satisfy $\phi'(t) = 0$ and $\phi''(t) \geq 0$. By (5.43) we deduce that $t^2 \leq \frac{N+2+4\sqrt{N-2}}{N-6}$ and a contradiction arises. That proves claim (iv).

Finally (v) follows from (i)-(iv). □

The following proposition allows us to reduce the existence of zeroes of σ to the existence of zeroes of a function which only depends on one variable.

Proposition 5.8. *Let* $\Phi : (0, +\infty) \to \mathbb{R}$ *be defined by*

$$\Phi(t) := t^{N-4}\left[a(0)B + b(0)\phi(t)\right] - \left[d(0)B + c(0)\phi\left(\frac{1}{t}\right)\right], \qquad (5.44)$$

where ϕ *is defined in* (5.38) *and* B *is given in Lemma* 4.2.

If t^* *is a stable zero of the function* Φ *with*

$$a(0)B + b(0)\phi(t^*) > 0, \qquad (5.45)$$

then $\lambda^* = (\lambda_1^*, \lambda_2^*)$ *with* $\lambda_1^* = \left\{2\left[(N-2)A^2\tau(0)\right]^{-1}\left[a(0)B + b(0)\phi(t^*)\right]\right\}^{\frac{1}{N-4}}$ *and* $\lambda_2^* = t^*\lambda_1^*$ *is a stable zero of the function* σ *defined in* (5.37).

Moreover if $\sigma(\lambda_1, \lambda_2) = 0$ *then* $\Phi(\lambda_2/\lambda_1) = 0$ *and condition* (5.45) *holds.*

Proof. It is enough to point out that $\sigma(\lambda_1, \lambda_2) = 0$ if and only if

$$\begin{cases} \frac{N-2}{2}A^2\tau(0)\lambda_1^{N-4} - a(0)B - b(0)\frac{1}{\lambda_1}\int\limits_{\mathbb{R}^N} \psi^0_{\lambda_1,0}U_{\lambda_2,0} = 0 \\ \frac{N-2}{2}A^2\tau(0)\lambda_2^{N-4} - d(0)B - c(0)\frac{1}{\lambda_2}\int\limits_{\mathbb{R}^N} \psi^0_{\lambda_2,0}U_{\lambda_1,0} = 0 \end{cases}$$

and if we choose $\lambda_2 = t\lambda_1$ the previous system is equivalent to

$$\begin{cases} \frac{N-2}{2} A^2 \tau(0) \lambda_1^{N-4} = a(0)B + b(0)\phi(t) \\ t^{N-4}\left[a(0)B + b(0)\phi(t)\right] - \left[d(0)B + c(0)\phi\left(\frac{1}{t}\right)\right] = 0. \end{cases}$$

The claim easily follows. □

Finally we can prove our second main result.

Proof of Theorem 1.4. To prove this theorem, taking into account Proposition 5.8 and Theorem 5.6, we will show that the function Φ has, in each case, a stable critical point that verifies (5.45)

Proof of (1). If $N \geq 7$, then by (v) of Lemma 5.7 we get $\Phi(\zeta) < 0 < \Phi\left(\frac{1}{\zeta}\right)$. Hence there exists $t^* \in \left(\zeta, \frac{1}{\zeta}\right)$ which is a stable zero of the function Φ and satisfies (5.45), because $b(0)\phi(t^*) > 0$.

If $N = 6$, then by (i) and (ii) of Lemma 5.7 $\lim_{t\to 0} \Phi(t) = -Cc(0) < 0$ and $\lim_{t\to+\infty} \Phi(t) = +\infty$. Hence there exists a stable zero t^* for the function Φ, that verifies (5.45).

An analogous argument proves the case $N = 5$.

Proof of (2). If $N \geq 7$, then by Lemma 5.7 we get

$$\lim_{t\to 0} \Phi(t) = -d(0)B < 0 \text{ and } \lim_{t\to+\infty} t^{-(N-4)}\Phi(t) = a(0)B > 0. \tag{5.46}$$

So there exists t^* which is a stable zero of the function Φ. We have to prove that t^* satisfies (5.45). In fact if $t^* > \zeta$ then $\phi(t^*) > 0$ and (5.45) holds. If $t^* < \zeta$ then $\phi\left(\frac{1}{t^*}\right) > 0$ and

$$a(0)B + b(0)\phi\left(t^*\right) = \frac{1}{t^{*N-4}}\left[d(0)B + c(0)\phi\left(\frac{1}{t^*}\right)\right] > 0, \tag{5.47}$$

namely (5.45) holds.

Taking into account Lemma 5.7, easier calculations prove the cases $N = 5, 6$.

Proof of (3). Assume $N \geq 7$. We get again (5.46), so a stable zero t^* for the function Φ exists. We have to verify that such a stable critical point verify also (5.45). If

$$K := \min_{(0,+\infty)} \left[d(0)B + c(0)\phi\left(\frac{1}{t}\right)\right] = \left[d(0)B + c(0)\phi\left(\frac{1}{m}\right)\right] > 0,$$

then (5.47) shows our claim. If $K \leq 0$, then $\Phi(m) > 0$, so we can take $t^* \in (0, m)$. Hence (5.45) follows since ϕ is decreasing in $(0, m)$. □

Using Proposition 5.8 and Theorem 5.6 we can give the following example.

Example 5.9. Assume $N \geq 7$, $a(0) > 0$ and $c(0) = 0$. Then there exists $d^* > 0$ and $b^* > 0$ such that for any $d(0) \in (0, d^*)$ and $|b(0)| > b^*$ there exists three different families of symmetric solutions of problem (1.1) that concentrates at the origin.

Remark 5.10. If $N = 5, 6$, then case (3) of Theorem 1.4 in general does not hold.

For example, if $N = 6$ and $a(0)B + b(0)C > 0$ and $d(0)B + c(0)C > 0$ (see Lemma 4.2 and (i) in Lemma 5.7) it is not difficult to see that the function Φ has a stable zero which satisfies (5.45). If $N = 5$ and $a(0) > 0$, $b(0) < 0$ and $c(0) < 0$ are fixed and $d(0) > 0$ is large enough, then there exists a stable zero for the function Φ which satisfies (5.45).

On the contrary if $N = 5, 6$ $a(0) > 0$, $b(0) < 0$ and $c(0) < 0$ are fixed and $d(0) > 0$ is small enough, then the function Φ has not a zero which satisfies (5.45).

Remark 5.11. Let Ω be symmetric with respect to the origin (i.e. $x \in \Omega$ iff $-x \in \Omega$) then it is easily seen that all the arguments of this section works, making use of the subspace of H given by

$$\tilde{\mathrm{H}}_s = \left\{(u_1, u_2) \in \mathrm{H} \ : \ u_i(x) = u_i(-x), \ \forall x \in \Omega_\varepsilon, \ i = 1, 2\right\}.$$

So, in this new setting, we can get results analogous to those referred in Theorem 1.4.

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